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Squaring the Circle and Leonardos Vitruvian man

Today I want to talk about

- A mathematical problem, which is about 3000 years old – squaring the Circle
- A 500 years old famous drawing – Leonardos Vitruvian man and
- A classroom experiment using new technological tools – Derive, Cabri, TI92

Two years ago, a friend of mine brought some informations supposing that I was interested.
Reading the paper I found this:

„Geometrical secret of Leonardo da Vinci discovered"
„Leonardos solution of squaring the circle"

My first reactions were:

„That´s impossible, we all know, the problem of squaring the circle has no solution!"
„The drawing is 500 years old and nobody discovered this before??"

My feelings wavered between doubting scepticism and „That´s unbelievable, I must check it!"
But my friend was right: **I was interested!!**
When in 1998 this book was published, I had a lot of informations to work with:
Klaus Schröer (a mathematic-artist) and Klaus Irle (an art historian) worked together and published this book:

Klaus Schröer/Klaus Irle
„Ich aber quadriere den Kreis ..."
Waxmann, Münster, 1998
ISBN 3-89325-555-9

Leonardo da Vinci

First of all I want to make some remarks on Leonardos life and work.
He was born in the year 1452 in **Vinci**, that is a little town near Florence in Italy. On invitation of the King of France, he spent the last three years of his life in **Amboise** in the valley of Loire. When he was travelling from Italy to France in the year 1516 he had in his luggage three paintings, one of it the famous **„Mona Lisa"**. That is the reason why nowadays you can see this painting in Paris (Louvre) and not in Italy.
Hundreds of researchers worked on Leonardos manuscripts and assets. He is one of the best investigated artist in the world. Nevertheless we still have surprising results.

Art and Mathematics

In the last 500 years nobody analysed the drawing under a mathematical point of view!
What is the reason?
Between the time of Renaissance and today we have a tremendous change in the conception of art. On Leonardos point of view, the concept of art is connected with rules, exercises and ingenuity. All artistic subjects can be explained with simple mathematical concepts. Today the development of art is associated with the absence of rational thinking. The artist expresses his feelings.
The view on the mathematical message of this drawing was covered with a smoke-screen in a intellectual climate, in which art and mathematics are mutually exclusive.

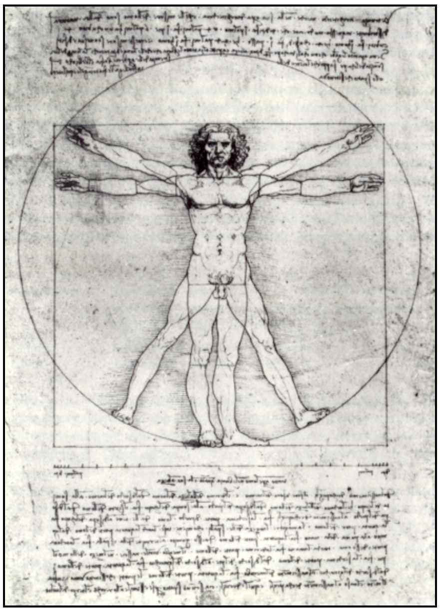
Squaring the Circle

Squaring the Circle is one of three great problems of Classical Geometry and in the Renaissance the most famous problem of Mathematics. For more than 3000 years

mathematicans have worked on the problem of constructing a square equal in area to that of a given circle.
Whether or not it is possible depends, of course, on what tools you allow yourself. **Plato** insisted that the problem be solved with **straightedge** and **compass** only.
Remark: When Lindemann proved in 1882 that Pi is transcedental he effectively proved that the solution is impossible in a finite number of constructions.
Leonardo expressed several times his intention to write a book about Geometry. In this book he intended to describe various procedures to solve the problem of squaring the circle. But this book was never finished!!

Vitruvian man

(Vitruv, Roman architect, wrote 30 B.C. the book „De Architectura“, which influenced architecture in the Renaissance)

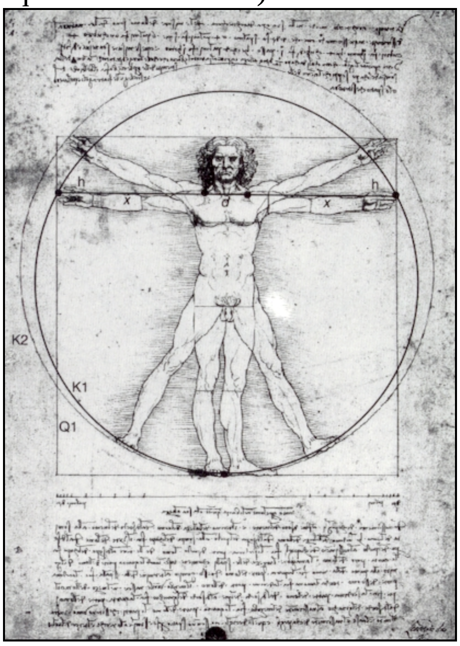


In the Internet you can find a lot of Web-Sites with Leonardos drawing, but nearly half of the pictures are **reversed**. Possibly the authors didn´t realize that Leonardo wrote all his texts in **mirror-writing**!!

In the classroom we made exact measurements and compared this with Leonardos text.
Comparing areas of the square and the circle (center is the man´s navel) you can easily see a difference.

Circle: 176.72 cm²
Square: 153.51 cm²

Schröer and Irle focussed there eyes on the **rotation of the arms**. The center of the rotation is marked in the drawing. (Ratio of length of the arms and length of the square is **fak=0.436**)



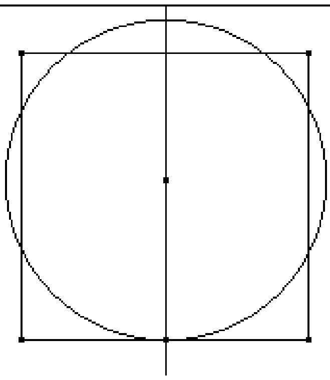
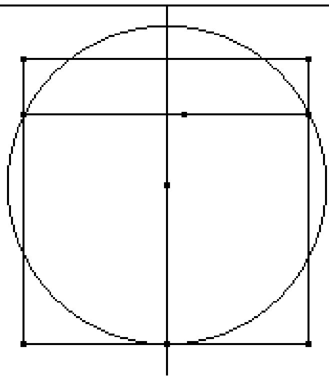
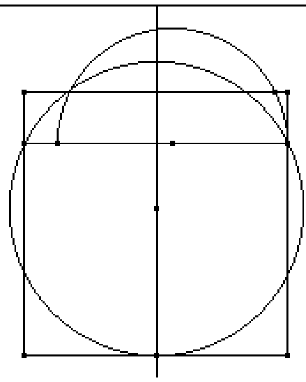
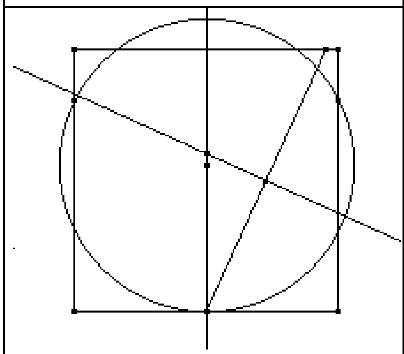
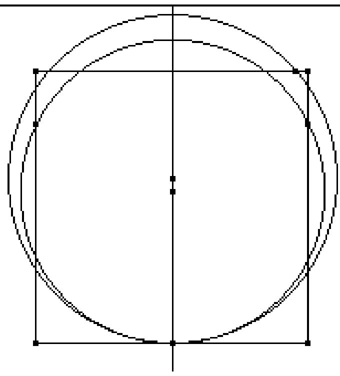
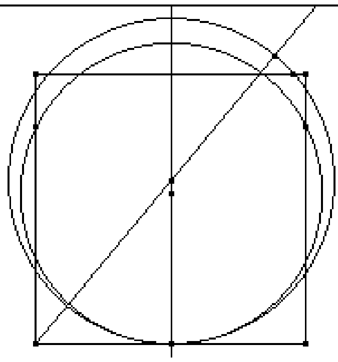
First of all they asked for a circle possibly equal in area with the given square. So they realizied that the circle you can easily construct has nearly the same area as the square!
All research workers didn´t pay attention to this circle !!

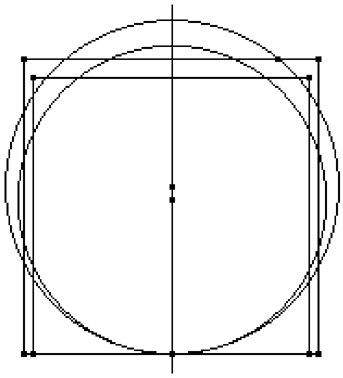
Circle: 153.94 cm²

But what´s about a second square equal in area with the larger circle?? (Leonardos drawing contains the characteristic of a so called **emblem**, which were used in the 16^{th} and 17^{th} century)
Schröer and Irle found the **missing link**:
Connect the corner of the square with the center of the circle and you can construct a larger square (nearly) equal in area to the circle.

Square: 176.89 cm²

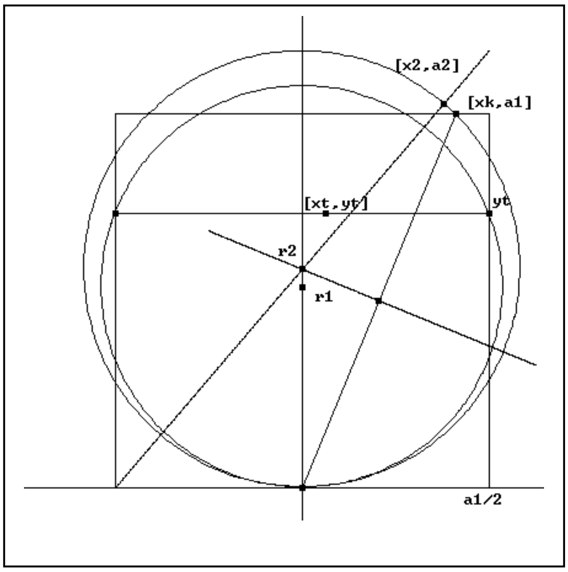
The real surprise is the procedure for a construction, defined in this way:

		
We start with a square and a circle intersecting the square’s sides	Connect upper intersection points, multiply length of the square with 0.436 and we get midpoint of the rotation of the arms	Intersection point of arm rotation with upper side of the square is marked.
		
Perpendicular bisector yields center point of the larger circle	Construct the larger circle	Connecting the edge of the square and center point of the larger circle and you can draw intersection point with the larger circle



Second square is drawn

First Calculations



In the classroom we started with an example:
(a1 = 10 and r1 = 5.4)
(Geometrical construction and algebraic calculations parallel)

- For the geometrical constructions (by hand) we used straightedge and compass. I think, it is necessary to use the fingers for at least one example!!
- For the calculations we used Derive.

You need equation of a straight line and equation of a circle in this form:

$$(x - x_m)^2 + (y - y_m)^2 = r^2$$

$$a1 := 10$$
$$r1 := 5.4$$
$$x^2 + (y - r1)^2 = r1^2$$
$$\left(\frac{a1}{2}\right)^2 + (y - r1)^2 = r1^2$$
$$[y = 7.43960,\ y = 3.36039]$$
$$yt := 7.4396$$
$$rt := a1 \cdot 0.436$$
$$xt := \frac{a1}{2} - rt$$
$$(x - xt)^2 + (y - yt)^2 = rt^2$$
$$(x - xt)^2 + (a1 - yt)^2 = rt^2$$
$$[x = 4.16901,\ x = -2.88901]$$
$$xk := 4.16901$$

$$\frac{y - \frac{a1}{2}}{x - \frac{xk}{2}} = -\frac{xk}{a1}$$
$$\frac{y - \frac{a1}{2}}{0 - \frac{xk}{2}} = -\frac{xk}{a1}$$
$$[y = 5.86903]$$
$$r2 := 5.86903$$
$$x^2 + (y - r2)^2 = r2^2$$
$$y = \frac{2 \cdot r2}{a1} \cdot x + r2$$
$$x^2 + \left(\frac{2 \cdot r2}{a1} \cdot x + r2 - r2\right)^2 = r2^2$$
$$[x = 3.80606,\ x = -3.80606]$$

$$x2 := 3.80606$$
$$y = \frac{2 \cdot r2}{a1} \cdot x2 + r2$$
$$y = 10.3366$$
$$a2 := 10.3366$$
$$a1^2 = 100$$
$$\pi \cdot r1^2 = 91.6088$$
$$a2^2 = 106.845$$
$$\pi \cdot r2^2 = 108.213$$
$$\frac{\pi \cdot r2^2}{a2^2} \cdot 100 = 101.280$$

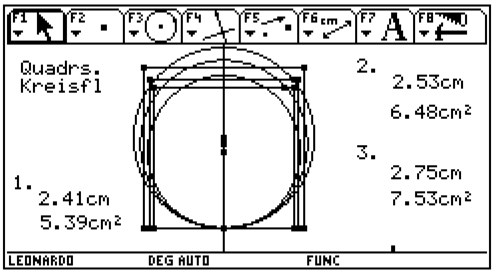
The role of the CAS:
We use Derive to solve 4 equations but the process must be **controlled by the students**.

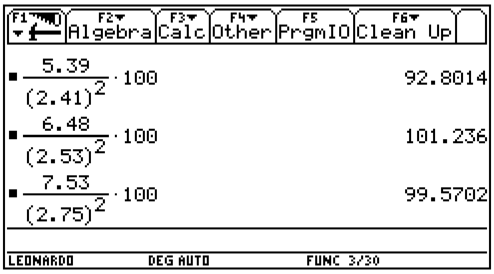
Questions

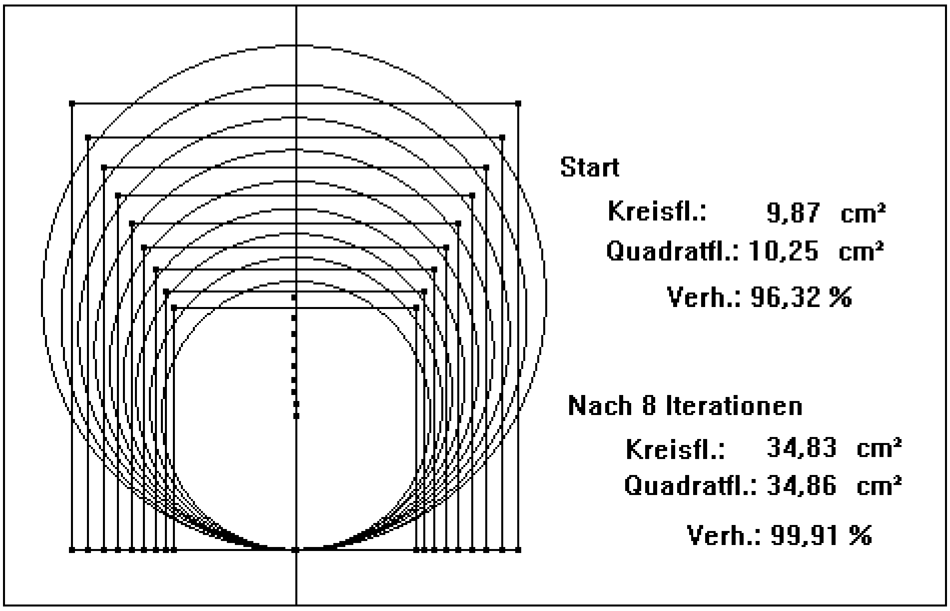
- Does the sequence of ratios of areas (circle – square) converge?
- What´s the limit of this sequence?
- What´s the effect of other dimensions of the initial pair (circle – square)?
- Does the sequence converge, if you use different factors (fak $\neq$ 0.436) ?
- What´s the effect of the factor on the limit of the sequence?

Geometrical Iteration with Cabri

One of the most powerful tools is the application of **macros**.
We use for simplification **fak:= 0.5**







Iterations with Derive

We cannot be satisfied with these results. Our next calculation (algebraic-analytical methods) uses the ITERATES-Function of Derive.

Syntax of the **ITERATES**-Function: **ITERATES(Operation , Variable , Initial Object , Number of Iterations)**

Informations in a row:

- Length of initial square **a1**
- Radius of initial circle **r1**
- (Calculated) length of final square **a2**
- (Calculated) Radius of final circle **r2**
- Area of final circle
- Area of final square
- **Ratio** of this areas in percent

Operation (generating a row by the preceeding row) :

Take the 3rd and 4th value in the following row as initial objects!!

fak := 0.436

$$\mathrm{ITERATES}\left(\left[a1 := z_3,\ r1 := z_4,\ a2,\ r2,\ \pi\cdot r2^2,\ a2^2,\ \frac{\pi\cdot r2^2}{a2^2}\cdot 100\right], z, \mathrm{startzeile}, 10\right)$$

$$\mathrm{startzeile} := \left[0,\ 0,\ 10,\ 5.4,\ \pi\cdot 5.4^2,\ 10^2,\ \frac{\pi\cdot 5.4^2}{10^2}\cdot 100\right]$$

0	0	10	5.4	91.6088	100	91.6088
10	5.4	10.3366	5.86903	108.214	106.845	101.280
10.3366	5.86903	11.1745	6.30184	124.762	124.870	99.9136
11.1745	6.30184	12.0253	6.78626	144.681	144.608	100.050
12.0253	6.78626	12.9470	7.30590	167.686	167.626	100.035
12.9470	7.30590	13.9387	7.86556	194.361	194.288	100.037
13.9387	7.86556	15.0064	8.46807	225.277	225.194	100.037

Different initial objects:

fak:= 0.436

a1:= 10	r1:= 5.2	r1:= 5.6	r1:= 6.0	r1:=8.0
	84.9486	98.5203	113.097	201.061
	103.204	100.204	99.2032	106.642
	99.7444	100.019	100.127	99.5
	100.068	100.039	100.027	100.094
	100.034	100.037	100.038	100.031
	100.037		100.037	100.037

Different factors:

fak:=0.35	fak:=0.45	fak:=0.46	fak:=0.5
84.9486	84.9486	84.9486	84.9486
105.305	103.021	102.902	102.500
99.7254	99.7364	99.7301	99.7029
100.351	100.034	100.012	99.9334
100.265	100.004	99.9845	99.9126
100.277	100.007	99.9873	99.9144
100.275		99.9871	99.9142
		99.9870	

Summing up

The special attraction is to work at **different levels**:

- Historical aspects
- Geometric-constructiv work with compass and straightedge
- Calculations of intersection-points (line – circle), lengths and ratios
- Constructions with a DGS
- Iterations with a CAS

> The real fascination is not answering the question for the problem of squaring the circle, but the convergence of that sequence generated by Leonardo´s procedure!!
>
> Thats the real secret of this drawing!!

Student´reactions

„ Most important for me was, that mathematics is not only a boring manipulation of terms , but can be an exiciting puzzle.“
„ I liked the variation“
„Most important for me was to be able to control the computer-applications for the solutions“
„I liked to discover a little bit of Math-history“
„The actuallity!“